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Plane-symmetric cosmological model II

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Abstract. Considering the cylindrically-symmetric metric of Marder a plane-symmetric cosmological model has been derived which is of class higher than one. Various physical and geometrical properties of the model have been discussed. It is found that when the model is filled with disordered radiation it becomes conformal to flat space–time and reduces to a particular case of the Lemaître universe.

1. Introduction

The general cylindrically-symmetric metric is considered in the form given by Marder (1958)

$$ds^2 = A^2(dt^2 - dx^2) - B^2\,dy^2 - C^2\,dz^2, \tag{1.1}$$

where A, B, C are functions of x and t. The necessary condition for the model to represent a perfect fluid distribution and the explicit expressions for the pressure and density are obtained in terms of the curvature components. Considering a special case of the above line element when the metric potentials B and C are equal and A, B are functions of t alone, a plane-symmetric perfect-fluid cosmological model has been derived which is not of class one. The explicit expressions for the pressure and density satisfying the reality conditions and the equation of state for the model have been obtained. When the model represents a universe filled with disordered radiation it becomes conformally flat and reduces to a particular case of the Lemaître universe. From the consideration of geodesic equations we infer that a particle at rest in the model remains permanently at rest. The expression for the generalized Doppler effect in the model has been given. Consideration of the geodesic deviation vector leads to the conclusion that the universe is expanding with time. Out of the fourteen scalar invariants of second order evaluated for this model only two are independent. The line element admits a four-parameter group of motions.

The eight algebraically independent and nonvanishing components of the curvature tensor R_{hijk} for the metric (1.1) are given by

$$R_{1212} \equiv f_1 = B\left(B_{11} - \frac{A_1B_1 + A_4B_4}{A}\right),$$

$$R_{1313} \equiv \bar{f}_1 = C\left(C_{11} - \frac{A_1C_1 + A_4C_4}{A}\right),$$

$$R_{1224} \equiv f_2 = B\left(\frac{A_1B_4 + A_4B_1}{A} - B_{14}\right),$$

$$R_{1334} \equiv \bar{f}_2 = C\left(\frac{A_1C_4+A_4C_1}{A}-C_{14}\right),$$

$$R_{2424} \equiv f_3 = B\left(B_{44}-\frac{A_1B_1+A_4B_4}{A}\right),$$

$$R_{3434} \equiv \bar{f}_3 = C\left(C_{44}-\frac{A_1C_1+A_4C_4}{A}\right),$$

$$R_{1414} \equiv f_4 = (A_1^2-A_4^2)+A(A_{44}-A_{11}),$$

$$R_{2323} \equiv f_5 = \frac{BC}{A^2}(B_1C_1-B_4C_4). \tag{1.2}$$

The suffixes 1 and 4 after the symbols A, B, C denote ordinary differentiation with respect to x and t respectively. The energy–momentum tensor for a perfect fluid distribution is given by

$$T_{ij} = (p+\epsilon)v_iv_j-pg_{ij} \tag{1.3}$$

together with

$$g_{ij}v^iv^j = 1,$$

p being the pressure, ϵ the density and $v_i = (v_1, 0, 0, v_4)$ the flow vector which represents the motion of the fluid in the x direction. The field equations

$$R_{ij}-\tfrac{1}{2}Rg_{ij} = -8\pi T_{ij} \qquad \text{(with } C = G = 1 \text{ and } \Lambda = 0\text{)}$$

for the line element (1.1) couched in terms of f's are as follows:

$$\frac{f_3}{B^2}+\frac{\bar{f}_3}{C^2}-\frac{A^2}{B^2C^2}f_5 = -8\pi\{(p+\epsilon)v_1^2+pA^2\}, \tag{1.4}$$

$$\frac{B^2}{A^4}f_4+\frac{B^2}{A^2C^2}(\bar{f}_3-\bar{f}_1) = -8\pi pB^2, \tag{1.5}$$

$$\frac{C^2}{A^4}f_4+\frac{C^2}{A^2B^2}(f_3-f_1) = -8\pi pC^2, \tag{1.6}$$

$$\frac{f_1}{B^2}+\frac{\bar{f}_1}{C^2}+\frac{A^2}{B^2C^2}f_5 = -8\pi\{(p+\epsilon)v_4^2-pA^2\}, \tag{1.7}$$

$$\frac{f_2}{B^2}+\frac{\bar{f}_2}{C^2} = 8\pi\{(p+\epsilon)v_1v_4\}. \tag{1.8}$$

From equations (1.5) and (1.6) we have

$$\frac{f_3-f_1}{\bar{f}_3-\bar{f}_1} = \frac{B^2}{C^2}. \tag{1.9}$$

Eliminating p, ϵ and v_i from equations (1.4) to (1.8) we get the necessary condition for a perfect fluid distribution as

$$\left(\frac{f_1}{B^2}+\frac{\bar{f}_3}{C^2}\right)^2 = \left(\frac{f_4}{A^2}+\frac{A^2}{B^2C^2}f_5\right)^2+\left(\frac{f_2}{B^2}+\frac{\bar{f}_2}{C^2}\right)^2. \tag{1.10}$$

In view of (1.10) the pressure and density are given by

$$8\pi p = \frac{f_1 - f_3}{A^2 B^2} - \frac{f_4}{A^4}, \tag{1.11}$$

$$8\pi \epsilon = \frac{f_3 - f_1}{A^2 B^2} - \frac{f_4}{A^4} - \frac{2f_5}{B^2 C^2}. \tag{1.12}$$

The velocity vector of the distribution is given by

$$v_i = \left\{ \left(\frac{F-1}{2} \right)^{1/2} A, 0, 0, \left(\frac{F+1}{2} \right)^{1/2} A \right\}, \tag{1.13}$$

where

$$F = \left(\frac{\bar{f}_3}{C^2} + \frac{f_1}{B^2} \right) \left(\frac{f_4}{A^2} + \frac{A^2}{B^2 C^2} f_5 \right)^{-1}. \tag{1.14}$$

The physical conditions $\epsilon > 3p > 0$ and the fact that $F \geqslant 1$ demand that

$$\frac{1}{2}\left(\frac{f_4}{A^4} - \frac{f_5}{B^2 C^2} \right) > \frac{f_1 - f_3}{A^2 B^2} > \frac{f_4}{A^4}$$

and

$$\frac{\bar{f}_3}{C^2} + \frac{f_1}{B^2} \geqslant \frac{f_4}{A^2} + \frac{A^2}{B^2 C^2} f_5. \tag{1.15}$$

2. A special case

A general solution for the field of the perfect fluid in the case of the metric (1.1) when A, B, C are functions of t alone has been given by Singh and Singh (1969). In order to find a specific solution we confine ourselves to the case when the metric (1.1) reduces to the form

$$ds^2 = A^2(dt^2 - dx^2) - B^2(dy^2 + dz^2), \tag{2.1}$$

A, B being functions of t alone. To reduce the number of arbitrary functions we further assume that

$$f_1 = f_3. \tag{2.2}$$

This gives

$$B = \alpha_0 t + \beta_0, \tag{2.3}$$

where α_0 and β_0 are constants of integration. The necessary condition in this case takes the form

$$\frac{2f_1}{B^2} = \frac{f_4}{A^2} + \frac{A^2}{B^4} f_5, \tag{2.4}$$

and the distribution is given by

$$8\pi p = -\frac{f_4}{A^4}, \tag{2.5}$$

$$8\pi\epsilon = -\frac{f_4}{A^4} - \frac{2f_5}{B^4}. \tag{2.6}$$

Using (2.3) in (2.4) and solving the resulting differential equation

$$\left(\frac{A_4}{A}\right)_{,4} + \frac{2\alpha_0}{\alpha_0 t + \beta_0}\left(\frac{A_4}{A}\right) = \left(\frac{\alpha_0}{\alpha_0 t + \beta_0}\right)^2, \tag{2.7}$$

we get

$$A = \frac{\alpha_0 t + \beta_0}{\gamma_0} \exp\left(\frac{\delta_0}{\alpha_0 t + \beta_0}\right), \tag{2.8}$$

where γ_0 and δ_0 are constants of integration. Consequently the line element (2.1) takes the form

$$ds^2 = \left\{\frac{\alpha_0 t + \beta_0}{\gamma_0} \exp\left(\frac{\delta_0}{\alpha_0 t + \beta_0}\right)\right\}^2 (dt^2 - dx^2) - (\alpha_0 t + \beta_0)^2 (dy^2 + dz^2). \tag{2.9}$$

The transformations

$$\frac{\alpha_0 t + \beta_0}{\gamma_0} = T,$$

$$\gamma_0(dy + dz) = (dY + dZ) \tag{2.10}$$

and

$$dx = d\bar{X}$$

reduce (2.9) to

$$ds^2 = T^2 e^{a/T}(b\, dT^2 - dX^2) - T^2(dY^2 + dZ^2), \tag{2.11}$$

where a and b are arbitrary positive constants.

3. Some physical features

The distribution in the model is given by

$$8\pi p = \frac{e^{-a/T}}{bT^4}\left(1 - \frac{a}{T}\right), \tag{3.1}$$

$$8\pi\epsilon = \frac{e^{-a/T}}{bT^4}\left(3 - \frac{a}{T}\right). \tag{3.2}$$

The reality conditions $\epsilon > 3p > 0$ lead to

$$T > a > 0. \tag{3.3}$$

The equation of state for the model is given by

$$8\pi a^4 b(\epsilon-p)^5 \exp\left(\frac{\epsilon-3p}{\epsilon-p}\right) = 2(\epsilon-3p)^4. \tag{3.4}$$

The flow vector v^i of the distribution is given by

$$\begin{aligned} &v^1 = v^2 = v^3 = v_1 = v_2 = v_3 = 0, \\ &v^4 = \frac{e^{-a/2T}}{b^{1/2}T}, \qquad v_4 = b^{1/2}T e^{a/2T}. \end{aligned} \tag{3.5}$$

The flow vector v^i satisfies the equations of the geodesics $v^i_{;j}v^j = 0$. Thus the lines of flow are geodesics. A semicolon preceding a suffix denotes covariant differentiation. From (3.1) and (3.2) it follows that when $a = 0$, $\epsilon = 3p$, which implies that the space–time is filled with disordered radiation.

The motion of a test particle in the model is governed by the geodesics given by

$$\begin{aligned} &\frac{d^2X}{ds^2} + 2\left(\frac{1}{T} - \frac{a}{2T^2}\right)\frac{dX}{ds}\frac{dT}{ds} = 0, \\ &\frac{d^2Y}{ds^2} + \frac{2}{T}\frac{dY}{ds}\frac{dT}{ds} = 0, \\ &\frac{d^2Z}{ds^2} + \frac{2}{T}\frac{dZ}{ds}\frac{dT}{ds} = 0, \\ &\frac{d^2T}{ds^2} + \left(\frac{1}{T} - \frac{a}{2T^2}\right)\left\{\frac{1}{b}\left(\frac{dX}{ds}\right)^2 + \left(\frac{dT}{ds}\right)^2\right\} + \frac{e^{-a/T}}{bT}\left\{\left(\frac{dY}{ds}\right)^2 + \left(\frac{dZ}{ds}\right)^2\right\} = 0. \end{aligned} \tag{3.6}$$

If a particle is initially at rest, that is, if

$$\frac{dX}{ds} = \frac{dY}{ds} = \frac{dZ}{ds} = 0,$$

we get

$$\frac{dT}{ds} = \frac{e^{-a/2T}}{b^{1/2}T}.$$

From the equations of the geodesics we conclude that for all such particles the components of spatial acceleration would vanish, namely,

$$\frac{d^2X}{ds^2} = \frac{d^2Y}{ds^2} = \frac{d^2Z}{ds^2} = 0,$$

and the particle would remain permanently at rest.

The track of a light pulse in the model (2.11) is obtained by setting

$$ds^2 = 0,$$

that is,

$$\frac{1}{b}\left(\frac{dX}{dT}\right)^2 + \frac{e^{-a/T}}{b}\left\{\left(\frac{dY}{dT}\right)^2 + \left(\frac{dZ}{dT}\right)^2\right\} = 1, \tag{3.7}$$

and for the case when the velocity is along the Z axis equation (3.7) gives

$$\frac{dZ}{dT} = \pm b^{1/2}\, e^{a/2T} = \pm\phi(T). \tag{3.8}$$

Hence the light pulse leaving a particle at (0, 0, Z) at time T_1 would arrive at the origin at a later time T_2 given by

$$\int_{T_1}^{T_2} \phi(T)\, dT = \int_0^Z dZ.$$

Following the method outlined by Tolman (1962) the red-shift in this case is given by

$$\frac{\lambda+\delta\lambda}{\lambda} = \frac{T\, e^{a/2T}(b^{1/2}\, e^{a/2T_1} + U_Z)}{e^{a/2T_2}(bT^2\, e^{a/T} - U^2)^{1/2}}, \tag{3.9}$$

where U is the velocity of the source at the time of emission and U_Z, the Z component of the velocity.

4. Some geometrical features

The metric (2.11) is plane symmetric as it admits the group of motions

$$\bar{Y} = Y + l,$$

$$\bar{Z} = Z + m,$$

and rotation about the x axis, that is,

$$\bar{Y} = Y\cos\theta - Z\sin\theta,$$

$$\bar{Z} = Y\sin\theta + Z\cos\theta.$$

In addition it also admits the motion

$$\bar{X} = X + n.$$

It has been verified that the metric (2.11) is not of class one, that is, the line element cannot be embedded in a five-flat, whereas the Lemaître universe and the plane-symmetric cosmological model of Singh and Singh (1968) are both of class one.

The surviving components of the conformal curvature tensor C_{hijk} for the line element (2.11) are given by

$$C_{1212} = C_{1313} = -\frac{1}{b}C_{2424} = -\frac{1}{b}C_{3434} = \frac{a}{6bT},$$

$$C_{1414} = \frac{a}{3T}e^{a/T}, \tag{4.1}$$

$$C_{2323} = -\frac{a}{3bT}e^{-a/T}.$$

It follows that the space–time given by (2.11) is in general not conformally flat. However, if $a = 0$, that is, when the model represents a universe filled with disordered radiation it becomes conformal to flat space–time and reduces to a particular case of the Lemaître universe.

The time-curves for the metric (2.11) are given by X = constant, Y = constant, Z = constant, so that the components of the unit tangent vector u^i are

$$\left(0, 0, 0, \frac{\mathrm{d}T}{\mathrm{d}s}\right), \qquad \text{that is} \qquad \left(0, 0, 0, \frac{\mathrm{e}^{-a/2T}}{b^{1/2}T}\right).$$

As the unit tangent vector satisfies the equations of the geodesics $u^i_{;j}u^j = 0$, the time-curves are geodesics.

The geodesic deviation vector η^i is given by

$$\frac{\delta^2\eta^h}{\delta s^2}+R^h_{ijk}\frac{\mathrm{d}X^i}{\mathrm{d}s}\frac{\mathrm{d}X^k}{\mathrm{d}s}\eta^j = 0, \tag{4.2}$$

where $\delta/\delta s$ stands for absolute differentiation. Equations (4.2) on integration lead to the following:

$$\eta^1 = \alpha_1\int\frac{\mathrm{e}^{-a/2T}}{T}\mathrm{d}T+\beta_1, \tag{4.3}$$

$$\eta^2 = \alpha_2\int\frac{\mathrm{e}^{a/2T}}{T}\mathrm{d}T+\beta_2, \tag{4.4}$$

$$\eta^3 = \alpha_3\int\frac{\mathrm{e}^{a/2T}}{T}\mathrm{d}T+\beta_3, \tag{4.5}$$

$$\eta^4 = \alpha_4\frac{\mathrm{e}^{-a/2T}}{T}\int T\,\mathrm{e}^{a/2T}\,\mathrm{d}T+\beta_4\frac{\mathrm{e}^{-a/2T}}{T}, \tag{4.6}$$

where α's and β's are constants of integration. Because η^i is orthogonal to the tangent vector we have $\eta^4 \equiv 0$, which gives $\alpha_4 = \beta_4 = 0$. Following the method discussed in an earlier paper by Singh and Singh (1968) it can be shown that the magnitude of the deviation vector is an increasing function of time from and after a certain time T. Hence the model (2.11) is an expanding universe model.

It is found that the fourteen scalar invariants of second order (Narlikar and Karmarkar 1949) for the model are expressible in terms of only two independent invariants, that is, the pressure and density.

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